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EXPANDING GRAPHS AND INVARIANT MEANS

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All the known explicit constructions of expander families are essentially obtained by considering a sequence of finite index normal subgroups $N_i \lhd \Gamma$, and taking the Cayley graphs of Γ/N_i w.r. to the projection of a *global* finite set of generators of Γ. For many of these examples (e.g. $\Gamma = SL_2(\mathbb{Z})$, $\Gamma/N_i \cong SL_2(\mathbb{F}_p)$), we present first constructions of new, different, sets of generators for the finite quotients, which make the Cayley graphs an expander family. An intrinsic connection between the expanding property and uniqueness of the Haar measure on an appropriate compact group, as an invariant mean, is established and used in the construction of such generators.

1. Introduction and Statement of the Main Results

The theory of expanding graphs has had some remarkable developments in the last two decades. Explicit constructions of expanders have been presented using deep mathematical tools such as Kazhdan's property (T), the Ramanujan conjecture (proved by Deligne), and Selberg's theorem (see [9] for a comprehensive exposition of the subject). Typically, the expander graphs so-obtained are Cayley graphs $X\langle G_i, S_i\rangle$ of finite groups $\{G_i\}$, where for each i S_i is a set of k generators for the group G_i (with k fixed). Moreover, there is a countable group Γ, generated by a (symmetric) set of k elements S, such that each G_i is a quotient of Γ, and S is mapped onto each S_i under the quotient map. A fundamental problem in this theory is to understand how rigid are those sets of generators, of the finite groups, which make the corresponding Cayley graphs an expander family:

Question (Lubotzky–Weiss [16]): Does there exist an infinite family of finite groups $\{G_i\}$, a fixed integer k, and two systems of generators for the G_i's: S_i, S_i' with $|S_i| = |S_i'| = k$, such that $X\langle G_i, S_i\rangle$ is an expander family while $X\left\langle G_i, S_i'\right\rangle$ is not?

If all the G_i's arise as above, as quotients of one finitely generated group Γ, then changing one set of global generators to another, clearly does not change the expanding property. The problem becomes subtle when we change "locally"

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systems of generators for each G_i independently. In the present paper we investigate this question and exhibit examples of new expanding sets of generators for known families of finite groups. Our systems of generators are "global" in some sense: they are obtained by projecting sets of elements from the group Γ. However, in Γ these sets generate subgroups of *infinite* index.

Our first construction involves a distinguished family of graphs, so called "Ramanujan" (see the definition preceding Theorem 4.3 below), which are obtained as Cayley graphs of quotients of one finitely generated group.

Theorem 1.1. *Let $\Gamma = \langle F \rangle$ be a group generated by the finite symmetric set F. Suppose that $\{N_i\}$ is a decreasing sequence of finite index normal subgroups, and that $X\langle \Gamma/N_i, F\rangle$ are Ramanujan graphs. If $\{e\} \neq H \triangleleft \Gamma$ is any normal, non-trivial subgroup, then:*

1. *There exists $M < \infty$ such that for all i one has $[\Gamma/N_i : HN_i/N_i] \leq M$.*
2. *There exists a finite set $F_1 \subseteq H$ such that the Cayley graphs $X\langle HN_i/N_i, F_1\rangle$ form an expander family.*

Theorem 1.1 is a special case of a general result proved in 4.3. Recall that the Ramanujan graphs constructed in [11] and [18] (see also [20]) can be obtained as in Theorem 1.1, where Γ is a free group. Theorem 1.1 therefore presents a rich family of new expanding generators for the quotients. We remark, however, that the proof of this result, as well as the others described in the sequel, is completely non-constructive.

The following is the 2-dimensional case of a result which we consider in 4.4:

Theorem 1.2. *Let $H < SL_2(\mathbb{Z})$ be a subgroup (not necessarily of finite index), which is normal in some congruence subgroup. Let λ_0 denote the bottom of the Laplacian spectrum on the Riemannian manifold $H\backslash\mathbb{H}$ (the quotient of the hyperbolic plane $\mathbb{H}$ by the H-action – recall that for any discrete subgroup H one has $0 \leq \lambda_0 \leq 0.25$). If $\lambda_0 < 0.21$ then:*

1. *There exists $M < \infty$ such that the projection of H to every $SL_2(\mathbb{F}_p)$ generates a subgroup, denoted $SL_2(\mathbb{F}_p)_H$, of index $\leq M$.*
2. *There exists a finite set $F \subseteq H$ such that the Cayley graphs $X\left\langle SL_2(\mathbb{F}_p)_H, F\right\rangle$ form an expander family (even though H itself may not be finitely generated).*

When $H = SL_2(\mathbb{Z})$ we have $\lambda_0 = 0$, and thus Theorem 1.2 amounts to the well known construction of expander graphs on $SL_2(\mathbb{Z})$-generators. The value 0.21 comes from the recent improvement of Selberg's theorem due to Luo–Rudnick–Sarnak [15]. It would be interesting to eliminate the normality assumption in Theorem 1.2, and we believe that this should be possible (we shall see that for statement 1 there, this assumption is indeed redundant). A proof of this conjecture will provide many more sets of expanding generators. In fact, one can show that then Selberg's conjecture, which replaces the value 0.21 above by 0.25, implies that "most" finite sets $F \subseteq SL_2(\mathbb{Z})$ have this expanding property; for instance, any set which does not generate a virtually cyclic subgroup and contains a unipotent matrix (this follows from a result of Beardon -cf. [21, Cor.2], and covers the question of $F = \begin{pmatrix} 1 & \pm 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pm 3 & 1 \end{pmatrix}$ raised in [10]).

The value $\lambda_0(H\backslash\mathbb{H})$ for discrete subgroups of $SL_2(\mathbb{R})$, or, more generally, $\lambda_0(H\backslash\mathbb{H}^n)$ for subgroups $H < SO(n,1)$, was extensively studied in relation to other asymptotic properties of the subgroup H (see e.g. [22] and the references therein). In particular, many examples of subgroups satisfying the assumption of Theorem 1.2 are known, and in Section 5 we shall describe an *explicit* one (with concrete matrices), thereby obtaining expanding generators for $PSL_2(\mathbb{F}_p), p \equiv 1_{\mathrm{mod}\,3}$, coming from $PSL_2(\mathbb{C})$, which *do not* generate a lattice (see Theorem 5.2 below). This explicit new construction was recently investigated numerically by J. Lafferty and D. Rockmore [14], and turns out to yield surprisingly good expanders (almost, but not as good, as Ramanujan graphs). The construction is also a special case of the following general result, which we prove in Section 5:

Theorem 1.3. *Suppose that $\Gamma = \langle F \rangle$ and $\{N_i\}$ are as in 1.1. Assume that $X\langle \Gamma/N_i, F\rangle$ form an expander family, and that $H \triangleleft \Gamma$ is a normal subgroup for which the group Γ/H is amenable. Then statements 1 and 2 in Theorem 1.1 hold, namely, there exists a finite set of elements in H whose projection to Γ/N_i yields an expander family of Cayley graphs (up to a uniformly bounded index M).*

The proofs of the above results depend heavily on functional analysis tools. Extreme points and the Krein–Milman theorem, weak and strong topologies, and above all, the notion of invariant mean, play a fundamental role in our approach. The striking work of Lubotzky–Phillips–Sarnak [11], [12] revealed a surprising connection between the Ruziewicz problem and the construction of expander graphs, although this connection was somewhat circumstantial. In this paper we shall link the two problems together, and use considerations regarding invariant means to obtain the above results. Our approach is to translate the expanding property of Γ/N_i, to one concerning the uniqueness of the Haar measure on the profinite completion $G = \varprojlim \Gamma/N_i$ as a Γ-invariant mean on $L^\infty(G)$, and prove this uniqueness under appropriate conditions. The uniqueness question itself is not treated here as an object, but only as a tool for the combinatorial purposes, although some conclusions about the former can be drawn. For instance, from the proof of Theorem 1.1 one can deduce that if $\Gamma < SO(3)$ is a free subgroup as constructed by Lubotzky–Phillips–Sarnak in [12], and $\{e\} \neq H \triangleleft \Gamma$ is *any* non-trivial normal subgroup, then the Lebesgue measure on the 2-sphere is the unique H-invariant, finitely additive measure, defined on all the Lebesgue measurable subsets (see also the remark proceeding Theorem 4.3).

The main auxiliary result, which connects the notion of invariant mean to the combinatorial questions, is presented in Theorem 2.4 below. It is proved in [28], while here, for completeness, we give an almost full proof (without explaining one technical point which appears there). Theorem 2.4 connects the problems of expander graphs and invariant means with another theme, motivated by the work of Lubotzky–Phillips–Sarnak, namely, "Hecke operators of group actions". In [28], which is complementary to the present paper, it is put in a more general perspective and the following is deduced:

Theorem. *Let Γ be a finitely generated group acting by measure preserving transformations on a probability measure space (X,μ) (we do not assume ergodicity or that μ is non-atomic). Then one, and exactly one, of the following holds:*

1. *There exists a subgroup $H<\Gamma$ of finite index which acts trivially on $L^\infty(X,\mu)$.*
2. *There exists a subgroup of infinite index $H<\Gamma$ such that as Γ-representations, $\ell^2(\Gamma/H)$ is weakly contained in $L^2(X,\mu)$ (see Definition 2.3 below).*

This result implies, on one hand, Alon–Boppana's well known theorem (cf. [9, 4.2.6]: $\liminf\lambda_1(X_{n,k})\geq 2\sqrt{k-1}$ for any infinite sequence of k-regular graphs. On the other hand, it shows that for any (say, symmetric) set S, of k measure preserving transformations of a non atomic probability measure space (X,μ), the corresponding averaging operator T_S, on the space $L_0^2(X,\mu)$ of zero mean L^2-functions, satisfies the norm inequality:

$$\|T_S\|\geq\frac{2\sqrt{k-1}}{k}$$

(The case where X is the 2-sphere and S consists of elements from $SO(3)$ was treated in [12], G. Pisier [23] proved it when $S\subset SU(N)$ and X is the complex unit N-sphere for any $2\leq N\in\mathbb{N}$).

The rest of the paper is organized as follows: in Section 2 we consider some preliminary facts related to means on $L^\infty(X,\mu)$ and weak containment of unitary representations, and prove the main auxiliary result -Theorem 2.4. In Section 3 we show how to connect questions about expanders to invariant means on the profinite completion of a directed sequence of finite groups, and in Section 4 prove the main results (including Theorems 1.1 and 1.2 above). Finally, in Section 5 we study co-amenable subgroups (Theorem 1.3 above), and illustrate our results by constructing a new example of an expander family.

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Added in Proof. Using the methods developed in this paper, we were recently able to remove the normality assumption in Theorem 1.3 (for the precise result see: "Expander graphs and amenable quotients", to appear in "Emerging applications of number theory", IMA Volumes in Mathematics, Ed.: D. Hejhal et al.). We still do not know, however, whether this assumption is necessary for the results of Section 4.

2. Preliminaries

Means on $L^\infty(X,\mu)$. Here and throughout the rest of the section, (X,μ) denotes a probability measure space, i.e. μ is a σ-additive probability measure, defined on a σ-algebra $\mathbb{B}$ of subsets of X (which we shall seldom mention explicitly, for brevity). For $1 \leq p \leq \infty$, the space $L^p(X,\mu)$ has its usual meaning. $L^\infty(X,\mu)^*$, the dual space of $L^\infty(X,\mu)$, is equipped with both the norm and weak-* topology, and has a natural ordering: if $m_1, m_2 \in L^\infty(X,\mu)^*$ then $m_1 \geq m_2$ if for every $0 \leq f \in L^\infty(X,\mu)$ one has $m_1(f) \geq m_2(f)$. A *mean* on $L^\infty(X,\mu)$ is a positive element in $L^\infty(X,\mu)^*$ with $m(1)=1$. Notice that by Alaoglu's theorem, the set of means on $L^\infty(X,\mu)$ is compact in the weak-* topology.

There is a natural bijection between the non-negative functionals, denoted $L^\infty(X,\mu)^*_+$, and finitely additive measures m defined on the measurable sets and continuous with respect to μ (i.e. $\mu(A)=0 \Rightarrow m(A)=0$). Indeed, if $\varphi \in L^\infty(X,\mu)^*$ then one defines $m_\varphi(A) = \varphi(1_A)$ and conversely, given such measure m, $\varphi_m(f) = \int f dm$ belongs to $L^\infty(X,\mu)^*_+$. By abuse of notation we shall fail to distinguish between the two objects and identify a measure with its corresponding functional.

Definition 2.1. Given two elements $m_1, m_2 \in L^\infty(X,\mu)^*_+$ define:

$$m_1 \wedge m_2(A) = \inf\{m_1(B) + m_2(A-B) | B \subseteq A,\ B \in \mathbb{B}\}.$$

It is easy to verify the additivity of $m_1 \wedge m_2$ (see e.g. [2, 2.2.1]). As it is clearly non negative (it may be zero, even if m_1, m_2 are not), and continuous with respect to μ, we have $m_1 \wedge m_2 \in L^\infty(X,\mu)^*_+$. Obviously, $m_1 \wedge m_2 \leq m_1, m_2$. In fact, $m_1 \wedge m_2$ is the largest element in $L^\infty(X,\mu)$ with this property.

Lemma 2.2. *For any two means m_1, m_2 the following are equivalent:*

1. $m_1 \wedge m_2 = 0$.
2. *For every $\varepsilon > 0$ there exists $A \subseteq X$ such that $m_1(A) > 1-\varepsilon$ and $m_2(A) < \varepsilon$.*
3. $\|m_1 - m_2\| = 2$ *(norm on $L^\infty(X,\mu)^*$).*

Furthermore, if $m_2 = \mu$ then 2 may be replaced by:

$2'$. *For every $\varepsilon > 0$ there exists $A \subseteq X$ such that $m_1(A) = 1$ and $\mu(A) < \varepsilon$.*

Proof. $1 \Rightarrow 2$: Follows from the definition.
$2 \Rightarrow 3$: Take $f = 1_A - 1_{X-A}$.
$3 \Rightarrow 1$: If $\|f\|_\infty = 1$ and $m_1 - m_2(f) > 2-\varepsilon$, take $A = \{x | f(x) > 0\}$.

Clearly $2'$ implies 2. Conversely, if $m_2 = \mu$, given $\varepsilon > 0$ take for every n a set A_n satisfying $m(A_n) > 1 - \frac{\varepsilon}{2^n}$ and $\mu(A_n) < \frac{\varepsilon}{2^n}$. Then the set $A = \cup A_n$ is as required. ∎

Weak containment of unitary representations. The notion of weak containment of unitary representations will be essential for our discussion:

Definition 2.3. Let Γ be a discrete (resp. locally compact) group, and $(\sigma, \mathcal{H}_\sigma)$, $(\pi, \mathcal{H}_\pi)$ (continuous) unitary representations of Γ. We say that σ is *weakly contained* in π, denoted $\sigma \prec \pi$, if for every finite set of vectors $v_1, \cdots, v_n \in \mathcal{H}_\sigma$, a finite (resp. compact) subset $F \subseteq \Gamma$, and $\varepsilon > 0$, there exist $u_1, \cdots, u_n \in \mathcal{H}_\pi$ such that for all $\gamma \in F$ and $1 \leq i, j \leq n$:

$$\left|\langle \sigma(g) v_i, v_j \rangle - \langle \pi(g) u_i, u_j \rangle\right| < \varepsilon \tag{1}$$

By a simple approximation argument, notice that to show $\sigma \prec \pi$ it is enough to fix a (say, orthonormal) base $\{v_i\}$ for $\mathcal{H}_\sigma$, and establish (1) when the finite set of vectors v_i is taken from this base.

The following theorem links invariant means and the notion of weak containment. It is fundamental for the results obtained in the sequel:

Theorem 2.4. *Let Γ be a countable group acting by measure preserving transformations on a probability space (X, μ). Assume that m is a mean on $L^\infty(X, \mu)$ with the following properties:*

1. *$m \wedge \mu = 0$.*
2. *$\forall \gamma \in \Gamma$ either $\gamma m = m$ or $\gamma m \wedge m = 0$.*

Let $S = Stab(m) = \{\gamma \in \Gamma | \gamma m = m\}$. Then $\ell^2(\Gamma/S) \prec L_0^2(X, \mu)$ as Γ-representations (where $L_0^2(X, \mu)$ denotes the natural representation on the space of zero mean L^2-functions and $\ell^2(\Gamma/S)$ is the regular representation on the coset space Γ/S).

As mentioned in the introduction, the proof of Theorem 2.4 will appear in [28]. However, for completeness, we outline the proof here, avoiding one technical point which is explained in details there.

Sketch of proof. Fix once and for all a sequence $\gamma_i \in \Gamma$ such that $\gamma_i S$ are representatives for the cosets γS. Denote by $\overline{\gamma}_i$ the coset $\gamma_i S$ and notice that the functions $1_{\overline{\gamma}_i}$ form an orthonormal base for $\ell^2(\Gamma/S)$. Now use the remark after Definition 2.3 for this base, namely, given a finite subset $F_1 \subseteq \Gamma$, $\varepsilon > 0$, and a finite set of functions $1_{\overline{\gamma}_i}$, we need to exhibit functions $\varphi_i \in L_0^2(X, \mu)$ such that for all i, j:

$$\left|\left\langle \gamma 1_{\overline{\gamma}_i}, 1_{\overline{\gamma}_j} \right\rangle - \langle \gamma \varphi_i, \varphi_j \rangle\right| < \varepsilon \quad \forall \gamma \in F_1$$

namely,

$$\left|\left\langle \gamma_j^{-1} \gamma \gamma_i 1_{\overline{e}}, 1_{\overline{e}} \right\rangle - \langle \gamma \varphi_i, \varphi_j \rangle\right| < \varepsilon \quad \forall \gamma \in F_1 \tag{2}$$

Notice that the left expression in (2) is 1 or 0, depending on whether $\gamma_j^{-1} \gamma \gamma_i$ is in S or not. Therefore, if we group together, in a large finite set $F \subseteq \Gamma$, all the elements in Γ of the form $\gamma_j^{-1} \gamma \gamma_i$ obtained above, it clearly suffices to find $\varphi \in L_0^2(X, \mu)$ such that:

$$|\langle \gamma \varphi, \varphi \rangle - 1| < \varepsilon \quad \text{for } \gamma \in F \cap S \tag{3}$$

and

$$(4) \qquad |\langle\gamma\varphi,\varphi\rangle| < \varepsilon \quad \text{for } \gamma \in F - S$$

For then, the functions $\varphi_i = \gamma_i\varphi$ satisfy (2). To find such φ we use the mean m given by 2.4 as follows: The natural embedding of $L^1(X,\mu)$ in $L^\infty(X,\mu)^* \cong L^1(X,\mu)^{**}$ has a dense image so there exists a net $f_\alpha \in L^1(X,\mu), f_\alpha \geq 0 \; \|f_\alpha\| = 1$, with $f_\alpha \to m$ (i.e. $\forall\varphi \in L^\infty(X,\mu)$: $\int f_\alpha\varphi d\mu \to m(\varphi)$). For $\gamma \in S$, $\gamma f_\alpha \to \gamma m = m$, so $f_\alpha - \gamma f_\alpha \to 0$ weakly. As weak and strong closures of convex sets coincide, we may replace the f_α by convex combinations h_α, such that $h_\alpha \to m$ and $\|h_\alpha - \gamma h_\alpha\| \to 0$ for all $\gamma \in S$ (again $h_\alpha \geq 0$, $\|h_\alpha\| = 1$). Notice that using $\mu \wedge m = 0$ and Lemma 2.2, we may assume that:

$$\mu\{x|h_\alpha(x) > 0\} \to 0$$

Also note that the assumption $\gamma m \wedge m = 0$ for $\gamma \notin S$ (or equivalently, by 2.2, $\|m - \gamma m\| = 2$) implies that for any such γ one has $\|h_\alpha - \gamma h_\alpha\| \to 2$. Therefore, by taking an appropriate $h \in \{h_\alpha\}$ we get, for any fixed finite set $F \subseteq \Gamma$ and $\varepsilon > 0$, a function $h \geq 0$ ($\|h\|_1 = 1$) which satisfies the following three properties:

$$(5) \qquad \mu\{x|h(x) > 0\} < \varepsilon$$

$$(6) \qquad \|h - \gamma h\|_1 < \varepsilon \quad \text{for } \gamma \in F \cap S$$

$$(7) \qquad \|h - \gamma h\|_1 > 2 - \varepsilon \quad \text{for } \gamma \in F - S$$

Our first candidate for a function satisfying (3) and (4) above is then $\varphi = \sqrt{h}$. It is not difficult to verify that (6) implies that $\|\varphi - \gamma\varphi\|_2 < 2\varepsilon$ and thus (3) holds. Similarly, one "should" deduce from (7) that φ and $\gamma\varphi$ are "almost" orthogonal and deduce (4). One problem is that we need the function φ to be actually in $L_0^2(X,\mu)$. This is solved using (5) which implies:

$$\int \varphi(x)d\mu(x) = \int 1_A(x) \cdot \varphi(x)d\mu(x) \leq \|1_A\|_2\|\varphi\|_2 \leq \sqrt{\varepsilon} \text{ (where } A = \{x|h(x) > 0\}).$$

Therefore we can "correct" φ by replacing it with $\varphi - \int\varphi \in L_0^2(X,\mu)$, and as $\int\varphi$ is small, the considerations above will not be affected. The second and more significant problem is that unfortunately, (7) does not imply that $\left\langle\sqrt{h}, \gamma\sqrt{h}\right\rangle$ is small for a general function h. However, it is easy to see that this implication does hold if $h = 1_B/\mu(B)$ (for any set B). Thus, to complete the proof one needs to consider the technical issue of obtaining an appropriate set B from a function h satisfying (5), (6) and (7). This is done in [28] using a "slicing" method of Namioka. ∎

Remark. The reader familiar with the Ruziewicz problem, has probably noticed that the proof above applies a similar argument, based on an idea of Namioka, to the one used in [25] when presenting the key ingredient to the solution of that problem. However, here we need more, namely, one seeks for arbitrary small sets which are "almost invariant" with respect to the action of some elements, and "almost disjoint" for the action of others.

3. Invariant Means and the Expanding Property

We begin by recalling the definition of the main object of this paper:

Definition 3.1. A finite k-regular graph $X = X(V,E)$ with a set V of n vertices is called an (n,k,ε) expander if for every subset $A \subseteq V$:

$$|\partial A| \geq \varepsilon \left(1 - \frac{|A|}{n}\right) |A|$$

where $\partial A = \{y \in V | d(y,A) = 1\}$ is the "boundary" of A (d the distance function on X). A family of finite k-regular graphs (k-fixed) is called an *expander family* if there is $\varepsilon > 0$ such that all of them are (n,k,ε) expanders.

Naturally, the notion of expander family is interesting for an infinite family of graphs. By counting arguments it is known (see [26]) that "most" k-regular graphs are $(n,k,\frac{1}{2})$ expanders (at least for $k \geq 5$), but explicit constructions are difficult, and all of them are essentially Cayley graphs of finite quotients of certain countable groups. If G is a finite group generated by a (symmetric) set F then the Cayley graph of G with respect to F will be denoted by $X\langle G,F\rangle$.

Let Γ be a countable group and $F \subseteq \Gamma$ a finite set. Let $\cdots < N_2 < N_1 < \Gamma$ be a decreasing sequence of finite index normal subgroups. The assumption of normality here is redundant, as one may consider "Shreier graphs" instead of Cayley ones, however conceptually it is easier to work with the latter. The following well known result enables one to investigate expanding properties of Cayley graphs using representation theoretic tools:

Theorem 3.2. *For any finite (symmetric) set $F \subseteq \Gamma$, and subgroups $N_i < \Gamma$ as above, the following conditions are equivalent:*

1. *There is $\varepsilon_1 > 0$ such that the Cayley graphs $X\langle \Gamma/N_i, F\rangle$ are $([\Gamma:N_i],|F|,\varepsilon_1)$ expanders.*
2. *There exists $\varepsilon_2 > 0$ such that for every representation:*

$$\ell_0^2(\Gamma/N_i) = \left\{ f \in \ell^2(\Gamma/N_i) | \int f = 0 \right\}$$

and every unit vector $v \in L_0^2(\Gamma/N_i)$, there exists $\gamma \in F$ such that $\|\gamma v - v\| \geq \varepsilon_2$.

For a proof see [9, 4.3.2]. It is important to notice that we do not assume in 3.2 that F generates Γ (in fact Γ may not be finitely generated). Observe also that if $i > j$ there is a Γ-embedding $\ell^2(\Gamma/N_j) \subseteq \ell^2(\Gamma/N_i)$ induced by the natural projection from Γ/N_i to Γ/N_j (we assume all the counting measures are normalized). Therefore, one may define naturally the increasing limit $\lim \ell^2(\Gamma/N_i)$, as well as $\lim \ell_0^2(\Gamma/N_i)$, the latter being as in Theorem 3.2 (we shall shortly give another description of these representations). Hence we see that condition 2 in 3.2 may be replaced by:

2'. There exists $\varepsilon_2 > 0$ such that for every unit vector $v \in \lim \ell_0^2(\Gamma/N_i)$, there exists $\gamma \in F$ such that $\|\gamma v - v\| \geq \varepsilon_2$.

Let $G = \varprojlim \Gamma/N_i$ be the profinite completion of the directed sequence Γ/N_i. Then G is a compact (totally disconnected) group, and we denote by μ its (normalized) Haar measure. Γ is embedded naturally and densely in G, and its (measure preserving) multiplication action induces a unitary representation on $L_0^2(G,\mu)$- the space of zero integral, square-integrable functions on G. It is easy to verify that $L_0^2(G,\mu)$ and $\lim \ell_0^2(\Gamma/N_i)$ are isomorphic Γ-representations, which shows that in condition 2' we may replace them. We then have:

Theorem 3.3. *Let Γ and the N_i's be as above. Then the following are equivalent:*

1. *There exists a finite subset $F \subseteq \Gamma$ such that $X\langle \Gamma/N_i, F\rangle$ form an expander family.*
2. *$1_\Gamma \not\prec L_0^2(G,\mu)$ as Γ-representations (1_Γ denotes the trivial Γ-representation).*
3. *μ-integration is the unique Γ-invariant mean on $L^\infty(G,\mu)$.*

Proof. $1 \Rightarrow 2$: Follows from 3.2 ($1 \Rightarrow 2'$) and the remark preceding 2'.
$2 \Rightarrow 1$: By definition, if $(\pi, \mathcal{H})$ is any Γ-representation with $1_\Gamma \not\prec \pi$, then there exists a finite set $F \subseteq \Gamma$ and $\varepsilon > 0$, such that for all $v \in \mathcal{H}$ with $\|v\| = 1$ there exists $\gamma \in F$ with $\|\pi(\gamma)v - v\| \geq \varepsilon$. Now use again 3.2 ($2' \Rightarrow 1$) replacing $\lim \ell_0^2(\Gamma/N_i)$ by $L_0^2(G,\mu)$.
$2 \Leftrightarrow 3$: This is a special case of the results of Rosenblatt [25] ($2 \Rightarrow 3$) and Schmidt [27] ($3 \Rightarrow 2$), which discuss general measure preserving actions of countable groups. ∎

In the sequel we shall only need the equivalence between 1 and 3 in Theorem 3.3. As the proof shows, the notion of weak containment plays an essential role in establishing this connection. We shall make a deeper use of Theorem 3.3 in the next section. Here we confine ourselves to showing how it unifies questions about uniqueness of invariant means with ones regarding expander graphs (thereby connecting some of the problems raised in [9, Ch.10]).

Proposition 3.4. *Let $\cdots \to G_2 \to G_1$ be an infinite directed sequence of finite groups and $G = \varprojlim G_i$ their inverse limit. Let μ be the Haar measure of G. Then the following are equivalent:*

1. *There exists some k and a choice of k generators $\{g_{1,i} \cdots g_{k,i}\}$ for each G_i, such that the Cayley graphs of the G_i's with respect to these generators form an expander family.*
2. *There exists a finitely generated group $\Gamma < G$ such that μ is the unique Γ-invariant mean on $L^\infty(G,\mu)$.*

As mentioned in the introduction, an intriguing question is whether an expander family of Cayley graphs has this property with respect to *every* choice of k generators for each group. There is no known example or counterexample, however, analogous to Proposition 3.4, we can put the question in another perspective:

Proposition 3.5. *Keeping the notation of 3.4, the following are equivalent:*

1. *For every k and any choices of k generators for each G_i, the corresponding Cayley graphs form an expander family.*
2. *For every finitely generated dense subgroup $\Gamma < G$, μ is the unique Γ-invariant mean on $L^\infty(G,\mu)$.*

Proof of 3.4. $1 \Rightarrow 2$. Let $\varepsilon > 0$ be such that all the k-regular Cayley graphs are (n,k,ε) expanders. For every i choose any k elements $g'_{1,i},\cdots,g'_{k,i} \in G\times\cdots\times G$ such that their projection to G_i identifies with $g_{1,i},\cdots,g_{k,i}$. Recalling that $G\times\cdots\times G$ is compact, we may take a subsequence of the sequence $(g'_{1,i},\cdots,g'_{k,i})$ which converges, say, to $(g_1,\cdots,g_k)\in G^k$. We claim that $\Gamma = \langle g_1,\cdots,g_k\rangle$ satisfies condition 2 in 3.4. Indeed, by Theorem 3.3 we need to show that $1_\Gamma \not\prec L^2_0(G,\mu)$ as Γ-representations, which is equivalent to the existence of $\varepsilon > 0$ such that for every i and a unit vector $v \in \ell^2_0(G_i)$, there exists $g \in \{g_1,\cdots,g_k\}$ such that $\|gv - v\| > \varepsilon$ (g acts through the natural projection from G to G_i). However, the existence of such ε follows from the assumption, Theorem 3.2, and the observation that for any fixed G_i, the projection of $g_1,\cdots,g_k$ to G_i identifies with the projection of some $g_{1,j},\cdots,g_{k,j}$ for $j \geq i$ large enough (so that $(g_{1,j},\cdots,g_{k,j})$ is "close" to $(g_1,\cdots,g_k)$). For that j, $X\left\langle G_j,(g_{1,j},\cdots,g_{k,j})\right\rangle$ is an (n,k,ε) expander and as $\ell^2(G_i)$ is embedded in $\ell^2(G_j)$, the same estimate holds when projecting to G_i.

$2 \Rightarrow 1$. If N_i denotes the kernel of the natural projection $\Gamma \to G_i$, we have that $G = \varprojlim \Gamma/N_i$, and the claim follows from Theorem 3.3 ($3 \Rightarrow 1$). ∎

The proof of Proposition 3.5 relies on similar ideas as in that of 3.4, and is omitted.

In Condition 2 of 3.4 the assumption that Γ is finitely generated can be weakened by assuming only that Γ is countable. This follows from the fact that if $1 \not\prec \pi$ as Γ-representations, then this is so already when restricting to an appropriate finitely generated subgroup of Γ. We do not know, however, if this assumption can be weakened further by assuming only that μ is the unique G-invariant mean, and this seems to be a rather subtle question. Notice that the action of G on $L^\infty(G,\mu)^*$ is not continuous (and not even measurable), thus invariance under a dense subgroup does not guarantee G-invariance. Let us give an example, based on one presented by Lubotzky and Weiss [16, 4.5]: For $G = \prod_{m\in\mathbb{N}} SL_n(\mathbb{Z}/m\mathbb{Z})$ it is shown there that G contains no finitely generated dense amenable subgroup, but it contains the dense amenable subgroup $\oplus SL_n(\mathbb{Z}/m\mathbb{Z})$. Notice that G has a dense finitely generated subgroup Γ for which $1_\Gamma \not\prec L^2_0(G,\mu)$, namely, $\Gamma = SL_n(\mathbb{Z})$ (for $n > 2$ this follows from property (T), while for $n = 2$ from Selberg's theorem, see [16]). Consequently, μ *is not unique* as an $\oplus SL_n(\mathbb{Z}/m\mathbb{Z})$-invariant mean, but it *is unique* as a G-invariant mean, as it is already unique for the action of $SL_n(\mathbb{Z})$. We do not know if there is a *finitely generated* dense subgroup for which μ is *not* unique. As shown in 3.5, this discussion could equivalently and equally well be given in the language of the expanding property, namely, the Cayley graphs of $SL_n(\mathbb{Z}/m\mathbb{Z})$ are expanders with respect to the projection of generators of $SL_n(\mathbb{Z})$, but are they expanders for any choice of k generators in each one separately? An interesting test case is the following question raised in [16]: If G is a compact group containing finitely generated dense subgroups A, B, such that A is amenable and B has property (T), is G necessarily finite? The challenging case, and the one relevant to our discussion, is when G is totally disconnected. As was shown, relaxing the finite generation assumption yields a negative answer.

Finally, although it diverges from our main discussion, we would like to use the opportunity to highlight the naturality and interest in questions regarding finitely additive invariant measures with another brief example. Recall that if $\Gamma(N) \subseteq \Gamma = SL_2(\mathbb{Z})$ denote the (principle) congruence subgroups, then Selberg's theorem asserts that $\lambda_1(\Gamma(N)\backslash\mathbb{H}) \geq \frac{3}{16}$ (where $\mathbb{H}$ denotes the upper half plane). This is a deep number theoretic result, and it would be of great interest to produce a more elementary proof for a "qualitative" Selberg theorem: $\lambda_1(\Gamma(N)\backslash\mathbb{H}) \geq c$ (for some, perhaps unknown, $c > 0$). Using Theorem 3.3 one can deduce:

Theorem 3.6. *Let* $\hat{\mathbb{Z}} = \prod_{p \text{ prime}} \mathbb{Z}_p$ *be the (compact) ring of adèlic integers. Then the following are equivalent:*

1. *A qualitative Selberg's theorem holds (i.e.,* $\exists c > 0$ *with* $\lambda_1(\Gamma(N)\backslash\mathbb{H}) \geq c > 0$*).*
2. *The Haar measure on* $SL_2(\hat{\mathbb{Z}})$ *is the unique* $SL_2(\mathbb{Z})$*-invariant, finitely additive measure, defined on all the Haar measurable subsets.*

Sketch of proof. It can be shown, using the non-amenability of $SL_2(\mathbb{Z})$, that any measure as in 2 of the theorem must be continuous with respect to the Haar measure on $SL_2(\hat{\mathbb{Z}})$ (see [9, Prop. 2.2.12]), and so it defines a mean on $L^\infty(SL_2(\hat{\mathbb{Z}}))$. Thus Theorem 3.3 may be applied, as soon as we know that $\lambda_1(\Gamma(N)\backslash\mathbb{H}) > c > 0$ is equivalent to $\Gamma/\Gamma(N)$ being an expander family. For this use Theorem 5.1 below. ∎

4. The Main Results

Our main results will be deduced from the following:

Theorem 4.1. *Let* Γ *be a countable group and* $\cdots < N_2 < N_1 < \Gamma$ *a decreasing sequence of finite index normal subgroups. Let* $H < \Gamma$ *be a subgroup (not necessarily finitely generated, or of finite index) with the following property: For every subgroup* $S < \Gamma$ *containing* H*, one has:*

$$\ell^2(\Gamma/S) \not\prec \oplus \ell_0^2(\Gamma/N_i) \tag{8}$$

as Γ*-representations. Then the following holds:*

1. *There exists* $M < \infty$ *such that for every* i *one has* $[\Gamma/N_i : HN_i/N_i] \leq M$.

If, moreover, H is normal in Γ, then the following holds as well:

2. *There exists a finite set $F \subseteq H$ such that its projection generates every HN_i/N_i, and the Cayley graphs $X_i = X\langle HN_i/N_i\,, F\rangle$ form an expander family.*

Proof. 1. Let $G = \varprojlim \Gamma/N_i$. The assertion is equivalent to the claim $[G:\overline{H}] < \infty$ ($\overline{H}$ being the closure of H in G, in its natural embedding). Assume the contrary, and let d be any G-invariant metric on G (compatible with the topology). Denote $S = \overline{H} \cap \Gamma$ and for every n set $A_n = \{x \in G \mid d(x, \overline{H}) < \frac{1}{n}\}$. Notice that for every $\gamma \in S$ one has $\gamma A_n = A_n$, whereas for any $\gamma \in \Gamma - S$: $\gamma A_n \cap A_n = \emptyset$, for all n with $\frac{1}{2n} < d(\gamma\overline{H}, \overline{H})$. For every n define the measure: $m_n(B) = \mu(B \cap A_n)/\mu(A_n)$, viewed as a mean on $L^\infty(G)$, and let m be any limit point of m_n (in the weak-* topology on $L^\infty(G,\mu)^*$). Since $\mu(A_n) \to 0$, $m \wedge \mu = 0$. Furthermore, for every $\gamma \in S$ we have $\gamma m_n = m_n$, so m is S-invariant. On the other hand, if $\gamma \in \Gamma - S$ there exists a set A_n such that $m(A_n) = 1$ and $\gamma m(A_n) = 0$, so $\gamma m \wedge m = 0$. Therefore, m satisfies the conditions of Theorem 2.4, while the conclusion of the latter contradicts (8), so 1 is established.

2. We first claim that in the assumption (8) one may replace Γ and the N_i's by $\Gamma' = \Gamma \cap \overline{H}$, $N_i' = N_i \cap \overline{H}$. Indeed, notice that $N_i = N_i'$ for all i large enough since from 1 it follows that $\overline{H}$ is open in G (and we may clearly assume that the N_i's have trivial intersection). As the N_i's are decreasing, the sum in the right hand side of (8) can be taken starting from any i, therefore, if there exists $H \leq S \leq \Gamma'$ with

$$\ell^2\left(\Gamma'/S\right) \prec \oplus \ell_0^2\left(\Gamma'/N_i'\right) = \oplus \ell_0^2\left(\Gamma'/N_i\right)$$

as Γ'-representations, we can induce the representations to Γ, thereby contradicting the assumption. Thus we assume hereafter that $\overline{H} = G \cong \varprojlim HN_i/N_i$ and by Theorem 3.3 we need to show that μ, the Haar measure of G, is the unique H-invariant mean on $L^\infty(G,\mu)$.

Consider the set M_H of H-invariant means on $L^\infty(G,\mu)$, which is a (weak*-) compact convex set. Let $Ext\, M_H$ denote the set of extreme points in M_H. Recall that a point x is said to be extreme if for every $y, z \in M_H$, and $0 < t < 1$ with $x = ty + (1-t)z$, necessarily $x = y = z$. From the definition it is clear that if $m \in Ext\, M_H$ and $m' \in L^\infty(G,\mu)^*$ is non-negative, H-invariant, and satisfies $m' \leq m$, then $m' = tm$ for some $0 \leq t \leq 1$. Therefore, recalling that $m_1 \wedge m_2 \leq m_1, m_2$ (and if m_1, m_2 are H-invariant then so is $m_1 \wedge m_2$), we deduce the following:

$(*)$ If $m_1, m_2 \in Ext\, M_H$ then $m_1 = m_2$ or $m_1 \wedge m_2 = 0$.

Now comes the only, but crucial, argument in which the normality of H in Γ is used. This assumption implies that M_H is invariant under the Γ action. (Indeed, if $m \in M_H$, $\gamma \in \Gamma$, $h \in H$ then $h\gamma m = \gamma h' m = \gamma m$, so γm is H-invariant as well). Furthermore, as the Γ-action preserves convex combinations, $Ext\, M_H$ is also Γ-invariant. Assume now that μ is not unique as an H-invariant mean on $L^\infty(G,\mu)$, so M_H has more than one point. By Krein–Milman's theorem M_H is the closure of the convex hull of $Ext\, M_H$, so in particular $Ext\, M_H$ must contain a mean $m \neq \mu$. We claim that m satisfies the two assumptions in Theorem 2.4 (with $(X,\mu) = (G,$ Haar measure)). Once this is verified, the proof will be complete, for Theorem 2.4 implies:

$$\ell^2(\Gamma/S) \prec L_0^2(G) \prec \oplus \ell_0^2(\Gamma/N_i)$$

$(H < S = Stab(m))$, contradicting the assumption (8).

Indeed, condition 2 in Theorem 2.4 holds by $(*)$ above and the fact that for every $\gamma \in \Gamma$, we have $\gamma m \in Ext\, M_H$. To show that condition 1 in Theorem 2.4 holds as well, it suffices, again by $(*)$, to prove that $\mu \in Ext\, M_H$ (recall $m \neq \mu$). This follows from the density of H in G or, more generally, from the ergodicity of its (multiplaication) action on G, as the following shows:

Lemma 4.2. *Suppose that the group H acts ergodically, by measure preserving transformations, on a probability measure space (X,μ) (i.e. every H-invariant measurable set is either null or has full measure, equivalently, every H-invariant $f \in L^1(X,\mu)$ is μ-essentially constant). Then μ is an extreme point in the (compact, convex) set M_H of H-invariant means on $L^\infty(X,\mu)$.*

Proof. Suppose $tm_1 + (1-t)m_2 = \mu$ with $0 < t < 1$, $m_1, m_2 \in M_H$. We need to show that $m_1 = \mu$ (similarly, this would show $m_2 = \mu$). Indeed, $m_1 \leq \mu/t$ so if $A_n \to \emptyset$ is a decreasing sequence with empty intersection, then $m_1(A_n) \to 0$. It follows that the additive measure m_1 is in fact σ-additive, and as it is continuous with respect to μ, it follows from the Radon–Nikodym theorem that there exists $f \in L^1(X,\mu)$ such that $m_1(\varphi) = \int \varphi f d\mu$. As m_1 is H-invariant, so is f, and by ergodicity $f \equiv c\mu$. Since everything is normalized, $c = 1$, which completes the proof. ∎

Remarks. 1. In contrast to the second part of Theorem 4.1, where we cannot do without the help of invariant means, this is not the case for part 1. A direct proof of 1 would use the sets A_n defined there, which already have all the required properties enabling one to "weakly embed" $\ell^2(\Gamma/S)$ in $L_0^2(G,\mu)$, exactly as in the proof of Theorem 2.4. To avoid repeating this argument we introduced the invariant mean, thereby applying directly 2.4.

2. We would like to emphasize the exact assumption needed instead of the normality of H: If Γ acts on (X,μ) and $H < \Gamma$ acts ergodically such that μ is *not* unique as an H-invariant mean on $L^\infty(X,\mu)$, then there exists an H-invariant mean m such that $m \wedge \mu = 0$, and for every $\gamma \in \Gamma$ either $\gamma m = m$, or $\gamma m \wedge m = 0$.

The proof of Theorem 4.1 shows that if H is normal in Γ then the assumption in the second remark above is always satisfied. A different approach to proving this assumption may also remove the use of extreme points, although this seems less probable. We remark that in [7] it is shown that if μ is not unique for an ergodic measure preserving action, then there are 2^c different invariant means (where c is the continuum cardinality), but the proof there does not seem to give the information we need.

We now turn to the results stated in the introduction. Recall that a (finite) k-regular graph is called *Ramanujan* if every eigenvalue λ of its adjacency matrix is either $\pm k$ or satisfies $|\lambda| \leq 2\sqrt{k-1}$. The value $\lambda = -k$ occurs exactly if X is bi-partite.. By a theorem of Alon–Boppana (which motivated the interest in this notion), Ramanujan graphs are those regular graphs which satisfy the strongest asymptotic bound on their eigenvalues. Such graphs (actually the only known infinite family of k-regular graphs) were constructed in [11] and [18] (and later, using similar methods, in [20]), essentially as quotients of free groups. These Ramanujan graphs were shown to share many extremal properties, to which we now contribute:

Theorem 4.3. *Let Γ be a countable group, and $N_i \triangleleft \Gamma$ a decreasing sequence of finite index normal subgroups. Let $H < \Gamma$ be a subgroup for which some $0 \leq f \in \ell^1(\Gamma)$ satisfies:*

$$\|f\|_{\ell^2(\Gamma/H)} > \|f\|_{\oplus \ell_0^2(\Gamma/N_i)}$$

(when the representations are extended naturally to $\ell^1(\Gamma)$). Then:

1. *There exists $M < \infty$ such that for all i one has $[\Gamma/N_i : HN_i/N_i] \leq M$.*

Furthermore, if $H \triangleleft \Gamma$ then:

2. *There exists a finite subset $F \subseteq H$, such that its projection generates each HN_i/N_i and $X\langle HN_i/N_i, F\rangle$ form an expander family.*

In particular, if $\Gamma = \langle F \rangle$ ($F \subseteq \Gamma$ finite and symmetric), and $X\langle \Gamma/N_i, F\rangle$ are Ramanujan graphs, then for every non-trivial $H \triangleleft \Gamma$ assertions 1 and 2 above hold.

Proof. By Theorem 4.1 it suffices to show that for every $H < S$ one has:

$$\ell^2(\Gamma/S) \not\prec \oplus \ell_0^2(\Gamma/N_i)$$

Assuming the contrary, we get for some S and every $f \in L^1(\Gamma)$:

$$\|f\|_{\ell^2(\Gamma/S)} \leq \|f\|_{\oplus \ell_0^2(\Gamma/N_i)} \tag{9}$$

(Actually, (9) is equivalent to a slightly weaker definition of weak containment than the one given in 2.1). In particular, this holds for every $0 \leq f$. However, for such f one has

$$\|f\|_{\ell^2(\Gamma/H)} \leq \|f\|_{\ell^2(\Gamma/S)} \tag{10}$$

This follows by considering the measure ν on Γ having f density with respect to the counting measure (normalized to be a probability measure), the interpretation of the norm of the convolution operator T_ν as the return probability of the ν-random walk, and the fact that Γ/H covers Γ/S. However, (9) and (10) together contradict the assumption.

To prove the second assertion regarding Ramanujan graphs, denote $|F| = k$ and consider $f = 1 + \frac{1}{k} \sum_{\gamma \in F} 1_\gamma$. The assumption that $X\langle \Gamma/N_i, F\rangle$ are Ramanujan implies:

$$\|f\|_{\oplus \ell_0^2(\Gamma/N_i)} \leq 1 + \frac{2\sqrt{k-1}}{k}$$

(although notice that if $X\langle\Gamma/N_i,F\rangle$ are bi-partite, it is not true that one necessarily has $\|f-1\|_{\oplus\ell_0^2(\Gamma/N_i)}\leq\frac{2\sqrt{k-1}}{k}$). By the first claim of the theorem (proved above) it therefore suffices to show that:

$$\|f\|_{\ell^2(\Gamma/H)}>1+\frac{2\sqrt{k-1}}{k} \tag{11}$$

Indeed, if (11) is not satisfied, then considering $\|f-1\|$ we would get, using the fact that on $\ell^2(\Gamma/H)$ one can always approximate the norm on *non-negative* functions in $\ell^2(\Gamma/H)$ (by replacing functions with their absolute value), that

$$\left\|\frac{1}{|F|}\sum_{\gamma\in F}1_\gamma\right\|_{\ell^2(\Gamma/H)}\leq\frac{2\sqrt{k-1}}{k}$$

Therefore the norm of $\frac{1}{|F|}\sum_{\gamma\in F}1_\gamma$ on $\ell^2(\Gamma/H)$ achieves the minimum possible already for the regular representation $\ell^2(\Gamma)$, and by Kesten's theorem (cf. [9, 4.5.3]) this implies that H must be trivial. ∎

Remark. Notice that we have actually established in the above proof the following: if $\Gamma=\langle F\rangle$ acts by measure preserving transformations on a probability space (X,μ), and the averaging operator on $L_0^2(X,\mu)$ defined by: $T_F(f)=\frac{1}{|F|}\sum_{g\in F}f\circ g$, satisfies $\|T_F\|=\frac{2\sqrt{k-1}}{k}$ $(k=|F|)$, then for any non trivial normal subgroup $H\triangleleft\Gamma$, μ is the unique H-invariant mean on $L^\infty(X,\mu)$. This may be applied to the free subgroups $\Gamma<SO(3)$, constructed in [12] (with X being the 2-sphere).

We now present a more geometrical condition, giving rise to new sets of generators e.g., for $SL_2(\mathbb{F}_p)$, which make the corresponding Cayley graphs an expander family:

Theorem 4.4. *Let $\mathbb{H}^n\cong SO(n,1)/SO(n)$ denote the n-dimensional real hyperbolic space. Suppose that $\Gamma<SO(n,1)$ is a lattice (i.e., a discrete subgroup of finite co-volume), $N_i\triangleleft\Gamma$ a decreasing sequence of finite index normal subgroups, and $H<\Gamma$ is any subgroup (not necessarily of finite index). Denote by $\lambda_1(N_i\backslash\mathbb{H}^n)$ the first non zero eigenvalue of the Laplacian Δ acting on the Riemannian manifold $N_i\backslash\mathbb{H}^n$, and by $\lambda_0(H\backslash\mathbb{H}^n)$ the bottom of its spectrum on $H\backslash\mathbb{H}^n$. If*

$$\lambda_0(H\backslash\mathbb{H}^n)<\inf\lambda_1(N_i\backslash\mathbb{H}^n)$$

then statement 1 of Theorem 4.1 holds. If, moreover, H is normal in Γ, then statement 2 there holds as well, namely, there exists a finite set of elements in H whose projection to Γ/N_i yields an expander family of Cayley graphs (up to a uniformly bounded index M).

Proof. By Theorem 4.1 it suffices to show that for every subgroup S of Γ, containing H, we have:

$$\ell^2(\Gamma/S) \not\prec \oplus\ell_0^2(\Gamma/N_i) \tag{12}$$

Indeed, if (12) does not hold for some $H \leq S \leq \Gamma$, then inducing both representations from Γ to $G = SO(n,1)$ yields:

$$L^2(G/S) \prec \oplus L_0^2(G/N_i) \tag{13}$$

as G-representations. Now consider only the class one (i.e. those with a $K = SO(n)$ invariant vector) irreducible representations, weakly contained in the representations of (13). By a well known result of Harish Chandra, these correspond, through the action of the Casimir operator, to the Laplacian spectrum, so from (13) we deduce that $\lambda_0(S\backslash\mathbb{H}^n) \geq \inf \lambda_1(N_i\backslash\mathbb{H}^n)$. We will therefore contradict the assumption of the theorem by showing that $\lambda_0(H\backslash\mathbb{H}^n) \geq \lambda_0(S\backslash\mathbb{H}^n)$. This, however, follows from the following general result: If $M \to N$ is a covering of Riemannian manifolds, then $\lambda_0(M) \geq \lambda_0(N)$. We sketch the proof, which is probably well known to experts: λ_0 can be characterized, on the one hand, as the bottom of the L^2-spectrum of Δ, but on the other, as the supremum of the "positive spectrum", i.e., the set of λ's for which there exists a *positive continuous* function f on the manifold, satisfying $\Delta f = \lambda f$ (see e.g. [29]). If $M \to N$ is a covering and $f: N \to \mathbb{R}^+$ satisfies $\Delta f = \lambda f$ then its lift to M has the same property, which shows that the supremum for M is at least as that for N, thereby proving our claim. ∎

Theorem 1.1, stated in the introduction, follows from Theorem 4.4 when considering $G = SL_2(\mathbb{R}) \cong SO(2,1)$, and using the recent improvement of Selberg's theorem due to Luo–Rudnick–Sarnak [15]: $\lambda_1(N_i\backslash\mathbb{H}) \geq 0.21$ for N_i the congruence subgroups of $SL_2(\mathbb{Z})$. Notice that by the Jacquet–Langlands correspondence, the same bound on λ_1 then holds for all the congruence subgroups of any arithmetic lattice in $SL_2(\mathbb{R})$ (see e.g. [9, 6.3.4]). A result similar to Theorem 4.4 is valid for the complex hyperbolic space and $G = SU(n,1)$, with exactly the same proof. We remark that it is convenient to take the lattice Γ in 4.4 to be arithmetic, and N_i as its congruence subgroups. In this case, good estimates are known for $\inf \lambda_1(N_i\backslash\mathbb{H}^n)$ (see e.g. [3] and the references therein). Notice that even if we relax the question of expanders, Theorem 4.4 may still be of interest, presenting a spectral condition for a subgroup of an arithmetic lattice to be congruence dense (and hence also Zariski dense).

Finally, we remark that an improvement of the bound on $\lambda_1(N_i\backslash\mathbb{H}^n)$ seemed to be conceptually related only to the quantitative size of the expanding coefficients of the resulting expanders. Theorem 4.4 shows that such improvement has also qualitative implications on the possibility to choose different sets of generators for the finite quotients, which make them an expander family.

5. Amenable Quotients and New Explicit Constructions

In this section we first prove Theorem 1.3 stated in the introduction. Notice that unlike the assumptions of Theorems 4.3 and 4.4, we do not impose in 1.3 any explicit norm estimate. The condition is more algebraic, and is easier to verify in many cases, so Theorem 1.3 is quite general in its nature. Following the proof of this theorem, we present an explicit new construction of expanding generators for $PSL_2(\mathbb{F}_p)\,(p \equiv 1_{\mathrm{mod}\,3})$, thereby illustrating both Theorems 4.4 and 1.3.

It is possible to deduce Theorem 1.3 directly from Theorem 4.1, however we would like to present a proof based on Theorem 4.4. For the proof we shall need:

Theorem 5.1. *Let G be a rank one simple Lie group with finite center, and $K < G$ a maximal compact subgroup. Let $\Gamma < G$ be a lattice, $\Gamma = \langle F \rangle$ ($|F| < \infty$), and $N_i \lhd \Gamma$ be as in Theorem 1.3. Then the following are equivalent:*

1. *$X\langle \Gamma/N_i, F\rangle$ is an expander family.*
2. *There exists $c > 0$ such that for all i one has $\lambda_1(N_i \backslash G/K) > c$.*

For Γ cocompact 5.1 follows from a general result of Brooks [4]. Partial results for noncompact Γ were considered by him in [5] (see also [6]). Brooks' approach is geometrical, while we present a short analytic proof which holds equally well for both the cocompact and non cocompact lattices:

Proof of 5.1. The correspondence between the Laplacian spectrum on $L^2(N_i \backslash G/K)$ and the class one G-representations (weakly contained) in $L^2(G/N_i)$, shows that assertion 2 is equivalent to:

2'. $$1_G \not\prec \oplus L_0^2(G/N_i)$$

Now, it is easy to see that the equivalence between 1 and 2' is a special case of the following result: If $\Gamma < G$ is as in Theorem 5.1, and π is any unitary Γ-representation, then one has $1_\Gamma \prec \pi$, if and only if $1_G \prec Ind_\Gamma^G \pi$ (where Ind denotes unitary induction). The implication $\Rightarrow$ in this assertion is simple, completely general, and follows from the "continuity of induction" in the Fell topology (see e.g. [9, 3.1.8]). The implication $\Leftarrow$ is more subtle and was proved by Margulis [19, III.1.11] under the assumption that $1_G \not\prec L_0^2(G/\Gamma)$ (it is not known whether this assumption is necessary). For lattices in semisimple Lie groups it is known that this assumption is always satisfied (see [1]), and this completes the proof. ∎

We can now turn to the

Proof of Theorem 1.3. The group Γ is an epimorphic image of a free group which may be taken as a lattice in $SL_2(\mathbb{R})$. Consider the inverse images of N_i and H under the assumed epimorphism, and notice that all the assumptions lift to the free group. Now observe that it is enough to deduce the assertion of 1.3 for the lifted subgroups of the free group, and then push it back to Γ, so, we may assume in the proof that Γ is already a free group which is a lattice in $SL_2(\mathbb{R})$.

Since $X\langle\Gamma/N_i,F\rangle$ is an expander family, it follows from Theorem 5.1 applied to $G=SL_2(\mathbb{R})$ that there exists $c>0$ such that $\lambda_1(N_i\backslash\mathbb{H})>c$ for all i. On the other hand, the amenability assumption on Γ/H implies that (and is in fact equivalent to) $\lambda_0(H\backslash\mathbb{H})=0$. Indeed, $1_\Gamma\prec\ell^2(\Gamma/H)$, so inducing from Γ to $G=SL_2(\mathbb{R})$ yields:

$$1_G\subseteq L^2(G/\Gamma)\prec L^2(G/H)$$

and $1_G\prec L^2(G/H)$ is equivalent to $\lambda_0(H\backslash\mathbb{H})=0$. We may now apply Theorem 4.4 to complete the proof. ∎

We now present an explicit new construction of expanding generators for $PSL_2(\mathbb{F}_p)$. For a numerical study of the graphs obtained here see [14].

Let $1\neq\omega\in\mathbb{C}$ be a primitive cube root of unity, and $PSL_2(\mathbb{Z}[\omega])$ the projective group of matrices with entries in the ring $\mathbb{Z}[\omega]$. $PSL_2(\mathbb{Z}[\omega])$ is an arithmetic non cocompact lattice in $PSL_2(\mathbb{C})\cong SO(3,1)$ (local isomorphism), therefore if $\{N_i\}$ is the family of its congruence subgroups, there is $c>0$ such that $\lambda_1(N_i\backslash\mathbb{H}^3)>c$. Recall that by Theorem 5.1 we then have that the (finite) congruence quotients form an expander family with respect to the projection of a global generating set.

Let Γ be the fundamental group of the complement of the figure 8 knot. The following facts are known (see [24] and [8]):

1. Γ has the presentation $\Gamma=\{a,b,x|xax^{-1}=ab^{-1},xbx^{-1}=b^2a^{-1}\}$.
2. Γ can be embedded as a subgroup of index 12 in $PSL_2(\mathbb{Z}[w])$ via the identifications:

$$a\mapsto\begin{pmatrix}1+\omega & -1\\ -\omega & 1\end{pmatrix}\quad b\mapsto\begin{pmatrix}-2\omega & -\omega\\ 1+\omega & 1+\omega\end{pmatrix}\quad x\mapsto\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$$

Henceforth we identify Γ with this embedding. Notice that a,b are free generators for a free, infinite index, normal subgroup $F_2\triangleleft\Gamma$, and that $\Gamma/F_2\cong\langle x\rangle\cong\mathbb{Z}$. Hence Γ/F_2 is amenable, which implies, as in the proof of Theorem 5.1, that $\lambda_0(F_2\backslash\mathbb{H}^3)=0$. Thus, applying either Theorem 4.4, or directly 5.1, we deduce that there is a finite set of elements in F_2 whose projection to the congruence quotients yields (up to a uniformly bounded index), an expander (Cayley-) graph family. Furthermore, as F_2 is generated by a and b, we may as well take them as our finite set of generators. A particularly convenient subfamily of the congruence quotients of $PSL_2(\mathbb{Z}[\omega])$ is the groups $PSL_2(\mathbb{F}_p)$, for those p for which a congruence projection $PSL_2(\mathbb{Z}[\omega])\twoheadrightarrow PSL_2(\mathbb{F}_p)$ exists, namely, the primes p such that the field $\mathbb{F}_p$ contains a primitive cube root of unity. As $\mathbb{F}_p^*$ is cyclic, such an element exists exactly if 3 divides $|\mathbb{F}_p^*|$ i.e., $p\equiv 1_{\mathrm{mod}\,3}$. Now, by Theorem 4.4 or 5.1 we know that the projection of F_2 to $PSL_2(\mathbb{F}_p)$ has a uniformly bounded index, however using the simplicity of $PSL_2(\mathbb{F}_p)$ and the normality of F_2 in Γ, it is easy to verify that for all $p\equiv 1_{\mathrm{mod}\,3}$ F_2 is mapped *onto* $PSL_2(\mathbb{F}_p)$. Thus, the above discussion amounts to:

Theorem 5.2. *Let $\omega \in \mathbb{C}$ be a primitive cube root of unity. Then* $F = \left\{ \begin{pmatrix} 1+\omega & -1 \\ -\omega & 1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} -2\omega & -\omega \\ 1+\omega & 1+\omega \end{pmatrix}^{\pm 1} \right\}$ *generates (freely) a free discrete subgroup of $PSL_2(\mathbb{C})$, which is of infinite index in $PSL_2(\mathbb{Z}[\omega])$. For all $p \equiv 1_{mod 3}$ the natural projection of F to $PSL_2(\mathbb{F}_p)$ generates the latter, and the graphs $X \langle PSL_2(\mathbb{F}_p), F \rangle$ form an expander family.*

As a different application of Theorems 4.4 and 1.3, consider the free group generated freely by the generators:

$$\Gamma = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$$

Γ is an arithmetic lattice in $SL_2(\mathbb{R})$, in fact it has index 6 in $SL_2(\mathbb{Z})$. The commutator subgroup $H = [\Gamma, \Gamma]$ has infinite index in Γ, however as in the previous example, Γ/H is abelian (hence amenable), and $\lambda_0(H\backslash\mathbb{H}) = 0$. Thus one can find there finitely many elements (in fact, commutators), which, up to a uniform finite index, make the Cayley graphs of the congruence quotients an expander family. The free subgroup $\Gamma_1 < \Gamma$ generated by this finite set of elements is no longer normal, and in fact, it can be shown that $\lambda_0(\Gamma_1\backslash\mathbb{H}) > 0$. Using Theorem 1.3 one may then continue this process with the free group Γ_1, and again go deeper to find a new set of expanding generators. Note that as before, since the groups $PSL_2(\mathbb{F}_p)$ are simple, we generate them in every stage. However, it is not clear how deep inside $SL_2(\mathbb{Z})$ one gets in this algorithm, again, as the proof is completely non-constructive.

References

[1] M. B. BEKKA: On uniqueness of invariant means, to appear in Proc. of AMS.

[2] K. P. S. BASHKARA RAO and M. BHASKARA RAO: *Theory of Charges,* Academic Press, 1983.

[3] M. BURGER and P. SARNAK: Ramanujan duals II, *Invent. Math.,* **106** (1991), 1–11.

[4] R. BROOKS: The spectral geometry of a tower of coverings, *J. of Diff. Geom.,* **23** (1986), 97–107.

[5] R. BROOKS: The spectral geometry of the Apollonian packing, *Comm. Pure. Appl. Math.,* **4** (1985), 359–366.

[6] T. COULHON and L. SALOFF-COSTE: Variétés riemanniennes isométrique á l'infini, *Ref. Mat. Iber.,* **11** (1995) no. 3, 687–726.

[7] C. CHOU: Ergodic group actions with non-unique invariant means, *Proc. A.M.S.,* **100** (1987), 647–650.

[8] J. HEMPEL: Coverings of Dehn fillings of surface bundles, *Topology and its applications,* **24** (1982), 157–170.

[9] A. LUBOTZKY: *Discrete Groups, Expanding Graphs and Invariant Measures,* Birkhauser, 1994.

[10] A. LUBOTZKY: Cayley graphs: eigenvalues, expanders and random walks, in: *Surveys in Combinatorics, 1995* (P. Rowbinson ed.), London Math. Soc. Lecture Note Ser. 218, Camb. Univ. Press, 1995, 155–189.

[11] A. LUBOTZKY, R. PHILLIPS and P. SARNAK: Ramanujan graphs, *Combinatorica,* **8** (1988), 261–277.

[12] A. LUBOTZKY, R. PHILLIPS and P. SARNAK: Hecke operators and distributing points on S^2,I, *Comm. Pure and Applied Math.,* **39** (1986), 149–186.

[13] J. LAFFERTY and D. ROCKMORE: Fast Fourier analysis for SL_2 over a finite field and related numerical experiments, *Experimental Mathematics,* **1** (1992), 45–139.

[14] J. LAFFERTY and D. ROCKMORE: Level spacings for $SL(2,p)$, To appear in "Emerging applications of number theory", IMA Volumes in Mathematics, ed: D. Hejhal et al.

[15] W. LUO, Z. RUDNICK and P. SARNAK: On Selberg's eigenvalue conjecture, *GAFA,* **5** (1995), No. 2, 387–401.

[16] A. LUBOTZKY and B. WEISS: Groups and expanders, in: "Expanding graphs" 95–109, DIMACS series Vol 10, American Math. Soc. 1993, (Ed. J. Friedman).

[17] G. A. MARGULIS: Explicit constructions of concentrators, *Prob. of Inform. Transf.,* **10** (1975), 325–332.

[18] G. A. MARGULIS: Explicit group theoretic constructions of combinatorial schemes and their applications for the construction of expanders and concentrators, *J. Prob. of Inform. Transf.,* **24** (1988), 39–46.

[19] G. A. MARGULIS: *Discrete Subgroups of Semisimple Lie Groups,* Springer-Verlag 1991.

[20] M. MORGENSTERN: Existence and explicit construction of $q+1$ regular Ramanujan graphs for every prime power q, *J. of Comb. Th. B,* **62** (1994), No. 1, 44–62.

[21] S. J. PATTERSON: The exponent of convergence of Poincare series, *Monat. f. Math.,* **82** (1976) 297–315.

[22] S. J. PATTERSON: Lectures on measures on limit sets of Kleinian groups, In: Analytical and Geometrical Aspects of Hyperbolic Space, Ed. D.B.A. Epstein, London Math. Soc. Lecture Notes Series 111, Cambridge Univ. Press, 1987.

[23] G. PISIER: Quadratic forms in unitary operators, preprint.

[24] R. RIGLEY: A quadratic parabolic group, *Proc. of the Camb. Phil. Soc.,* **77** (1975), 281–288.

[25] J. ROSENBLATT: Uniqueness of invariant means for measure preserving transformations, *Trans. AMS,* **265** (1981), 623–636.

[26] P. SARNAK: Some Applications of Modular Forms, *Cambridge Tracts in Mathematics,* **99**, Camb. Univ. Press 1990.

[27] K. SCHMIDT: Amenability, Kazhdan's property (T), strong ergodicity and invariant means for ergodic group actions, *Ergodic Th. and Dynamical Sys.,* **1** 223–236 (1981).

[28] Y. SHALOM: Hecke operators of group actions and weak containment of unitary representations, in preparation.

[29] D. SULLIVAN: *Related aspects of positivity* in: Collection: Aspects of mathematics and its applications, North-Holland Math. (1986) 747–779.

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